1. For the following functions, determine the nature of the singularity at $z = z_0$ (i.e. regular point, pole or essential singularity), compute the residue, calculate the Laurent series and determine the radius of convergence.

(a) $f(z) = \frac{z^2+2z+1}{z^2-1}$, $z_0 = 1$.

(b) $f(z) = z^2 \sin(\frac{1}{z})$, $z_0 = 0$.

(c) $f(z) = e^{\frac{1}{z}} \sin(\frac{1}{z})$, $z_0 = 0$.

2. Consider the function
$$f(z) = \frac{\sin(z^2+1)}{(z^2+1)^2}.$$

(a) Find all the singularities of f and determine their nature.

(b) Compute the residue at each singularity.

(c) Determine the radius of convergence of the Laurent series around each singularity.

3. Consider the function
$$f(z) = \frac{\sin(z)}{(z+1)(z-2)(z^2+1)}.$$

(a) Find all the singularities of f and determine the order of the poles.

(b) Let $\gamma(t) = 10e^{it}$, $t \in [0, 2\pi]$. Compute the integral
$$\int_\gamma f(z)\, dz.$$

4. Consider the function
$$f(z) = \frac{1}{z^4-1}.$$

(a) Find all the singularities of f and determine their nature.

(b) Compute the integral
$$\int_{\gamma_r} f(z)\, dz$$
for any value of $r \neq 1$, where γ_r is the circle of radius r centered at the origin and oriented counter-clockwise.

5. For the following functions, compute the *singular* part of their Laurent series at $z = z_0$ and determine the radius of convergence of the (full) Laurent series.

(a) $f(z) = \dfrac{\sin(z)}{\sin(z^2)}$, $z_0 = 0$.

(b) $f(z) = \dfrac{1}{\cos(\frac{\pi}{2}z)}$, $z_0 = 1$.

(c) $f(z) = \dfrac{\log(1+z)}{\sin(z^2)}$, $z_0 = 0$.

(d) $f(z) = \dfrac{\sin(z)}{z \cdot (e^z - 1)}$, $z_0 = 0$.

6. *(The nature of essential singularities.)* In this exercise, we will consider the function $f(z) = e^{\frac{1}{z}}$.

(a) Show that $f(z)$ has an essential singularity at $z_0 = 0$.

(b) Show that in any neighborhood of $z_0 = 0$, f attains every value of $\mathbb{C}^*$ infinitely often. More formally, you have to show the following: For any $R > 0$ and any $y \in \mathbb{C}$, there exist infinitely many z's with $|z| < R$ such that $f(z) = y$. *(Hint: If w_0 satisfies $e^{w_0} = y$, what are the other w's satisfying $e^w = y$?)*

Remark. In general, if a function f has an essential singularity at $z = z_0$, then in any neighborhood of z_0 the function will attain all values in $\mathbb{C}$ with the exception of at most one, infinitely many times. This is known as Picard's great theorem.

(c) Show that the above cannot be true for a function $g(z)$ with a pole at $z = 0$: In this case, show that, for $R > 0$, the function g restricted on the disc $B_R(0)$ only takes values satisfying $|g(z)| \geqslant R_1(R)$ with $R_1(R) \to +\infty$ as $R \to 0$.

Solutions

1. (a)

$$f(z) = \frac{z^2 + 2z + 1}{z^2 - 1}, z_0 = 1$$

1. Nature of the singularity:
Factorize the denominator:

$$z^2 - 1 = (z-1)(z+1)$$

At $z_0 = 1$, the factor $(z-1)$ appears in the denominator, but not in the numerator. Note, in particular, that $(z-1)$ appears to the first power in the denominator. In particular, we can express f as

$$f(z) = \frac{\left(\frac{z^2+2z+1}{z+1}\right)}{z-1} = \frac{g(z)}{z-1},$$

where the function $g(z) = \frac{(z^2+2z+1)}{z+1} = z+1$ is holomorphic at $z=1$. Hence, we calculate

$$\lim_{z\to 1}\big((z-1)\cdot f(z)\big) = \lim_{z\to 1} g(z) = g(1) = 2 \neq 0.$$

Therefore, it is a simple pole (that is to say, a pole of order 1).
2. Residue: For a simple pole at $z=z_0$, the residue is given by:

$$\mathrm{Res}_{z_0}(f) = \lim_{z\to z_0}(z-z_0)f(z)$$

Here, $z_0 = 1$, and we computed this integral above:

$$\mathrm{Res}_1(f) = \lim_{z\to 1}(z-1)\frac{z^2+2z+1}{(z-1)(z+1)} = 2.$$

3. Laurent series: Since $z_0 = 1$ is a simple pole, the Laurent series around $z_0 = 1$ has the form:

$$f(z) = \frac{a_{-1}}{z-1} + a_0 + a_1(z-1) + a_2(z-1)^2 + \dots$$

We already found $a_{-1} = 2$. To find the other coefficients, we can rewrite $f(z)$ in terms of $(z-1)$:

$$f(z) = \frac{(z+1)^2}{(z-1)(z+1)} = \frac{z+1}{z-1}$$

Expand $\frac{z+1}{z-1}$ around $z=1$:

$$\frac{z+1}{z-1} = \frac{(z-1)+2}{z-1} = 1 + \frac{2}{z-1}$$

Thus, the Laurent series is:

$$f(z) = \frac{2}{z-1} + 1$$

4. Radius of convergence: The Laurent series has radius of convergence $R=2$, since the next singularity is at $z=-1$, which is 2 units away from $z=1$.

(b) $f(z) = z^2 \sin\left(\frac{1}{z}\right)$, $z_0 = 0$
1. Nature of the singularity: Consider the series expansion of $\sin\left(\frac{1}{z}\right)$ (see also Exercise 3.a from the previous week):

$$\sin\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!z^{2n+1}}$$

Multiplying by z^2:

$$f(z) = z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!z^{2n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!z^{2n-1}}$$

This series has infinitely many negative powers of z, indicating that $z_0 = 0$ is an essential singularity.
2. Residue: The residue is the coefficient of $\frac{1}{z}$ in the Laurent series:

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! z^{2n-1}}$$

The term $\frac{1}{z}$ corresponds to $2n - 1 = 1$, i.e., $n = 1$. Therefore, the residue is:

$$\text{Res}_0(f) = \frac{(-1)^1}{3!} = -\frac{1}{6}.$$

3. Laurent series: The Laurent series is already derived:

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! z^{2n-1}}$$

4. Radius of convergence: The Laurent series converges for all $z \neq 0$, since there are no other singularities. Therefore, the radius of convergence is ∞.

(c)

$$f(z) = e^{\frac{1}{z}} \sin\left(\frac{1}{z}\right), z_0 = 0$$

Using the identity

$$\sin(w) = \frac{e^{iw} - e^{-iw}}{2i},$$

we can reexpress the above as

$$f(z) = \frac{1}{2i}\Big(e^{(1+i)\frac{1}{z}} - e^{(1-i)\frac{1}{z}}\Big).$$

1. Nature of the singularity: Consider the series expansions:

$$e^{\frac{1+i}{z}} = \sum_{n=0}^{\infty} \frac{(1+i)^n}{n! z^n}, \quad e^{\frac{1-i}{z}} = \sum_{n=0}^{\infty} \frac{(1-i)^n}{n! z^n}.$$

Subtracting the two, we get

$$f(z) = \frac{1}{2i}\Big(e^{(1+i)\frac{1}{z}} - e^{(1-i)\frac{1}{z}}\Big) = \sum_{n=0}^{\infty} \frac{\frac{1}{2i}\big((1+i)^n - (1-i)^n\big)}{n!} \frac{1}{z^n}.$$

Using the fact that

$$(1+i)^n - (1-i)^n = (1+i)^n\Big(1 - \big(\frac{1-i}{1+i}\big)^n\Big) = (1+i)^n(1 - (-i)^n),$$

we can readily verify that the coefficients of the above series are non zero except in the case when $n = 4k$, so we have a series with infinitely many negative powers of z, indicating that $z_0 = 0$ is an essential singularity.

2. Residue: The residue is the coefficient of $\frac{1}{z}$ in the Laurent series. From the above computation, we therefore get for $n = 1$:

$$\mathrm{Res}_0(f) = \frac{\frac{1}{2i}\big((1+i)^1 - (1-i)^1\big)}{1!} = 1.$$

3. We computed the Laurent series.

4. Radius of convergence: The Laurent series converges for all $z \neq 0$, since there are no other singularities. Therefore, the radius of convergence is ∞.

2. (a) 1. Singularities: The denominator $(z^2+1)^2$ has zeros at $z^2+1=0$, i.e., $z = \pm i$. These are the singularities of f.

2. Nature of the singularities: Let us expand the numerator as a series: Since

$$\sin(w) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} w^{2n+1}$$

for any w (the above power series has infinite radius of convergence), plugging in $w = z^2+1$ we get:

$$\sin(z^2+1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(z^2+1)^{2n+1} = (z^2+1) - \frac{1}{6}(z^2+1)^3 + \dots.$$

Therefore,

$$f(z) = \frac{(z^2+1) - \frac{1}{6}(z^2+1)^3 + \dots}{(z^2+1)^2} = \frac{1 - \frac{1}{6}(z^2+1)^2 + \dots}{z^2+1}.$$

From the above, we see that, since $z^2+1 = (z-i)(z+i)$, at $z = i$ the denominator has a simple root, while the numerator doesn't vanish; similarly at $z = -i$. Hence, $z = \pm i$ are simple poles (i.e. of order 1):

$$\lim_{z\to i}(z-i)f(z) = \lim_{z\to i}\left[(z-i)\frac{1-\frac{1}{6}(z^2+1)^2+\dots}{(z-i)(z+i)}\right]\lim_{z\to i}\frac{1-\frac{1}{6}(z^2+1)^2+\dots}{z+i} = \frac{1}{2i} \neq 0$$

and, similarly

$$\lim_{z\to -i}(z+i)f(z) = \lim_{z\to -i}\left[(z+i)\frac{1-\frac{1}{6}(z^2+1)^2+\dots}{(z-i)(z+i)}\right]\lim_{z\to -i}\frac{1-\frac{1}{6}(z^2+1)^2+\dots}{z-i} = -\frac{1}{2i} \neq 0$$

(b) 1. Residue at $z = i$: For a pole of order 1, the residue is given by:

$$\mathrm{Res}_{z_0}(f) = \lim_{z\to z_0}\left((z-z_0)f(z)\right).$$

For $z_0 = \pm i$, we calculated these limits above. So

$$\mathrm{Res}_i(f) = \frac{1}{2i}, \quad \mathrm{Res}_{-i}(f) = -\frac{1}{2i}.$$

(c) 1. Radius of convergence: The Laurent series at z_0 converges in the set $B_R(z_0) \setminus \{z_0\}$, where R is the distance from z_0 of the nearest point where f is not holomorphic. Since the singularities are at $z = \pm i$, the radius of convergence around each singularity is the distance to the other singularity, which is 2.

3. (a) The denominator $(z+1)(z-2)(z^2+1)$ has zeros at $z=-1$, $z=2$, and $z=\pm i$. These are the singularities of f.

Each factor appears to be the first power in the denominator, while the numerator $\sin(z)$ doesn't vanish at these points. So each singularity is a simple pole. We can calculate:

$$\lim_{z\to -1}\left[(z+1)\cdot\frac{\sin(z)}{(z+1)(z-2)(z^2+1)}\right] = \lim_{z\to -1}\left[\frac{\sin(z)}{(z-2)(z^2+1)}\right] = -\frac{1}{2}\sin(1) \neq 0,$$

$$\lim_{z\to 2}\left[(z-2)\cdot\frac{\sin(z)}{(z+1)(z-2)(z^2+1)}\right] = \lim_{z\to 2}\left[\frac{\sin(z)}{(z+1)(z^2+1)}\right] = \frac{1}{15}\sin(2) \neq 0,$$

$$\begin{aligned}\lim_{z\to i}\left[(z-i)\cdot\frac{\sin(z)}{(z+1)(z-2)(z-i)(z+i)}\right] &= \lim_{z\to i}\left[\frac{\sin(z)}{(z+1)(z-2)(z+i)}\right] \\ &= \frac{1}{2i(i+1)(i-2)}\sin(i) = \frac{1}{2(i+1)(i-2)}\sinh(1) \neq 0,\end{aligned}$$

$$\begin{aligned}\lim_{z\to -i}\left[(z+i)\cdot\frac{\sin(z)}{(z+1)(z-2)(z-i)(z+i)}\right] &= \lim_{z\to -i}\left[\frac{\sin(z)}{(z+1)(z-2)(z-i)}\right] \\ &= -\frac{1}{2i(i-1)(i+2)}\sin(-i) = \frac{1}{2(i-1)(i+2)}\sinh(1) \neq\end{aligned}$$

(b) 1. Integral: The contour $\gamma(t) = 10e^{it}$ is a circle of radius 10 centered at the origin. The singularities inside this contour are $z=-1$, $z=2$, and $z=\pm i$. By the Residue Theorem:

$$\int_\gamma f(z)\,dz = 2\pi i\left(\mathrm{Res}_{-1}(f) + \mathrm{Res}_2(f) + \mathrm{Res}_i(f) + \mathrm{Res}_{-i}(f)\right).$$

Compute each residue using the formula for simple poles:

$$\mathrm{Res}(f, z_0) = \lim_{z\to z_0}(z-z_0)f(z).$$

These are precisely the limits we calculated above, so

$$\int_\gamma f(z)\,dz = 2\pi i\left(-\frac{1}{2}\sin(1) + \frac{1}{15}\sin(2) + \frac{1}{2}\Big(\frac{1}{(1+i)(i-2)} + \frac{1}{(i-1)(i+2)}\Big)\sinh(1)\right).$$

4. (a) Find all the singularities of f and determine their nature.

1. Singularities: The denominator $z^4 - 1 = (z^2-1)(z^2+1) = (z-1)(z+1)(z-i)(z+i)$ has zeros at $z = \pm 1$ and $z = \pm i$. These are the singularities of f.

2. Nature of the singularities: Each factor appears to the first power in the denominator, while the numerator doesn't vanish anywhere. So each singularity is a simple pole (the corresponding limit $\lim_{z\to z_0}(z-z_0)f(z)$ exists and is non-zero).

(b) Compute the integral $\int_{\gamma_r} f(z)\,dz$ for any value of $r \neq 1$, where γ_r is the circle of radius r centered at the origin and oriented counter-clockwise.

1. Integral: The contour γ_r is a circle of radius r centered at the origin. The singularities inside this contour depend on r:

* If $r < 1$, there are no singularities inside the contour.
* If $r > 1$, the singularities inside the contour are $z = \pm 1$ and $z = \pm i$. By the Residue Theorem:

$$\int_{\gamma_r} f(z)\,dz = 2\pi i\left(\mathrm{Res}_1(f) + \mathrm{Res}_{-1}(f) + \mathrm{Res}_i(f) + \mathrm{Res}_{-i}(f)\right)$$

Compute each residue using the formula for simple poles:

$$\mathrm{Res}_{z_0}(f) = \lim_{z\to z_0}(z-z_0)f(z).$$

So we have:

$$\mathrm{Res}_1(f) = \lim_{z\to 1}\left[(z-1)\frac{1}{(z-1)(z+1)(z-i)(z+i)}\right] == \lim_{z\to 1}\frac{1}{(z+1)(z-i)(z+i)} = \frac{1}{4}$$

and, similarly:

$$\mathrm{Res}_{-1}(f) = -\frac{1}{4}, \quad \mathrm{Res}_i(f) = \frac{i}{4}, \quad \mathrm{Res}_{-i}(f) = -\frac{i}{4}.$$

So, in this case:

$$\int_{\gamma_r} f(z)\,dz = 0$$

* If $r =$, the integral is not well defined, as the singularities lie on the curve.

Remark. In the case of a holomorphic function that satisfies $Q(z) = Q(-z)$ (like z^4-1 in the above exercise; if Q is a polynomial, this holds exactly when it only has even powers of z), we have that, if z_0 is a root of Q, then so is $-z_0$. From the definition of the limit giving as the residue, it is easy to see that, if z_0 is a simple root of Q, then

$$\mathrm{Res}_{-z_0}\left(\frac{1}{Q(z)}\right) = -\mathrm{Res}_{z_0}\left(\frac{1}{Q(z)}\right).$$

Similarly, if $\overline{Q(z)} = Q(\bar{z})$ (if Q is polynomial, this happens exactly when Q has real coefficients), we can similarly easily check that

$$\mathrm{Res}_{\bar{z}_0}(\frac{1}{Q(z)}) = \overline{\mathrm{Res}_{z_0}(\frac{1}{Q(z)})}.$$

The above symmetries can sometimes help us avoid repetitive calculations.

5. (a)

$$f(z) = \frac{\sin(z)}{\sin(z^2)}, z_0 = 0$$

1. Singular part: The Taylor series for $\sin(z)$ around $z = 0$ is:

$$\sin(z) = z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots$$

The Taylor series for $\sin(z^2)$ around $z = 0$ is:

$$\sin(z^2) = z^2 - \frac{z^6}{6} + \frac{z^{10}}{120} - \cdots$$

Therefore, the function $f(z)$ can be written as:

$$f(z) = \frac{z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots}{z^2 - \frac{z^6}{6} + \frac{z^{10}}{120} - \cdots} = \frac{z}{z^2} \cdot \frac{1 - \frac{z^2}{6} + \frac{z^9}{120} - \dots}{1 - \frac{z^4}{6} + \frac{z^8}{120} - \dots} = \frac{1}{z} \cdot \frac{1 - \frac{z^2}{6} + \dots}{1 - \frac{z^4}{6} + \dots}.$$

Therefore, $z \cdot f(z)$ has a regular point at $z = 0$:

$$\lim_{z\to 0} \big(z \cdot f(z)\big) = \lim_{z\to 0} \frac{1 - \frac{z^2}{6} + \dots}{1 - \frac{z^4}{6} + \dots} = 1 \neq 0.$$

So f has a pole of order 1 at $z = 0$. This means that the Laurent series is of the form

$$f(z) = \frac{c_{-1}}{z} + c_0 + c_1 z + \dots,$$

so the singular part has only one term: $\frac{c_{-1}}{z}$. The coefficient c_{-1} is exactly what we computed above:

$$c_{-1} = \lim_{z\to 0} \big(z \cdot f(z)\big) = 1.$$

2. Radius of convergence: The radius of convergence of the Laurent series is determined by the distance to the nearest singularity other than $z = 0$. The next singularity of $\sin(z^2)$ is at $z = \pm\sqrt{\pi}$, which is approximately ± 1.77. Therefore, the radius of convergence is:

$$\sqrt{\pi} = 1.77\dots.$$

(b)

$$f(z) = \frac{1}{\cos\left(\frac{\pi}{2}z\right)}, z_0 = 1$$

1. Singular part: The above function has a pole of order 1 at $z_0 = 1$ so, like before, the singular part consists of only one term. Let us first verify that indeed we have a simple pole at $z_0 = 1$: We have

$$\begin{aligned}\cos(\frac{\pi}{2}z) &= \cos\left(\frac{\pi}{2}(z-1) + \frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}(z-1)\right)\\ &= -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(\frac{\pi}{2})^{2n+1}(z-1)^{2n+1}\\ &= \frac{\pi}{2}(z-1) - \frac{1}{6}\frac{\pi^3}{8}(z-1)^3 + \dots.\end{aligned}$$

Therefore, the function $f(z)$ can be written as:

$$f(z) = \frac{1}{\cos\left(\frac{\pi}{2}z\right)} = \frac{1}{\frac{\pi}{2}(z-1) - \frac{1}{6}\frac{\pi^3}{8}(z-1)^3 + \dots} = \frac{1}{z-1} \cdot \frac{1}{\frac{\pi}{2} - \frac{1}{6}\frac{\pi^3}{8}(z-1)^2 + \dots}.$$

Thus,

$$\lim_{z\to 1}(z-1)f(z) = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi} \neq 0,$$

so f has a pole of order 1 at $z_0 = 1$. The singular part is simply $\frac{c_{-1}}{z}$, with $c_{-1} = \frac{2}{\pi}$ (in view of the limit we just calculated).

2. Radius of convergence: The radius of convergence of the Laurent series is determined by the distance to the nearest singularity other than $z = 1$. The next singularity of $\cos\left(\frac{\pi}{2}z\right)$ is at $z = 3$. Therefore, the radius of convergence is 2.

(c)

$$f(z) = \frac{\log(1+z)}{\sin(z^2)}, z_0 = 0$$

1. Singular part: The Taylor series for $\log(1+z)$ around $z = 0$ is:

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$$

The Taylor series for $\sin(z^2)$ around $z = 0$ is:

$$\sin(z^2) = z^2 - \frac{z^6}{6} + \cdots$$

Therefore, the function $f(z)$ can be written as:

$$f(z) = \frac{z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots}{z^2 - \frac{z^6}{6} + \cdots} = \frac{1}{z} \cdot \frac{1 - \frac{z}{2} + \frac{z^2}{3} - \dots}{1 - \frac{z^4}{6} + \dots}.$$

So, f has a pole of order 1 at 0:

$$\lim_{z\to 0} zf(z) = 1.$$

Like before, the singular part of the Laurent series consists of only one term: $\frac{1}{z}$.
2. Radius of convergence: The radius of convergence of the Laurent series is determined by the distance to the nearest other point where f is not holomorphic. The nearest other roots of $\sin(z^2)$ are $\pm\sqrt{\pi}$, but note also that the function $\log(1+z)$ is holomorphic only on $\mathbb{C}\setminus(-\infty,-1]$. Hence, the radius of convergence is equal to 1 (the distance to $z=-1$).

(d)

$$f(z) = \frac{\sin(z)}{z(e^z-1)}, z_0 = 0$$

1. Singular part: The Taylor series for $\sin(z)$ around $z=0$ is:

$$\sin(z) = z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots$$

The Taylor series for $e^z - 1$ around $z=0$ is:

$$e^z - 1 = z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots$$

Therefore, the function $f(z)$ can be written as:

$$f(z) = \frac{z - \frac{z^3}{6} + \cdots}{z(z + \frac{z^2}{2} + \cdots)} = \frac{1 - \frac{z^2}{6} + \cdots}{z + \frac{z^2}{2} + \cdots} = \frac{1}{z}\cdot\frac{1 - \frac{z^2}{6} + \cdots}{1 + \frac{z}{2} + \cdots}.$$

Therefore,

$$\lim_{z\to 0} z\cdot f(z) = 1,$$

so f has a pole of order 1 and $c_{-1} = 1$. Thus, the singular part of the Laurent series is: $\frac{1}{z}$.
2. Radius of convergence: The radius of convergence of the Laurent series is determined by the distance to the nearest singularity other than $z=0$. The next singularities of e^z-1 are at $z = \pm 2\pi i$. Therefore, the radius of convergence is 2π.

6. **(a)** Essential singularity: The Laurent series of $e^{\frac{1}{z}}$ around $z=0$ is:

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!z^n}$$

This series has infinitely many negative powers of z, so $z=0$ is an essential singularity.

(b) We will show that, for any $R>0$ and any $y\in\mathbb{C}^*$, there exist infinitely many points z in $B_R(0)$ solving $e^{\frac{1}{z}} = y$. First of all, such a y, let $w = \log(y)$, so that $e^w = y$. For any $n\in\mathbb{Z}$, the points

$$w_n = w + 2\pi n i$$

also satisfy $e^{w_n} = e^{w+2\pi ni} = e^w = y$. Let us set

$$z_n = \frac{1}{w_n} = \frac{1}{2 + 2\pi ni}$$

Note that

$$e^{\frac{1}{z_n}} = y.$$

Moreover,

$$\lim_{n\to+\infty} z_n \lim_{n\to+\infty} \frac{1}{w + 2\pi ni} = 0.$$

Therefore, for any $R > 0$, there exists a $n_0 \geqslant 1$ such that, for any $n \geqslant n_0$, the point z_n satisfies $|z_n| < R$. Thus, we have infinitely many solutions z_n to the equation $e^{\frac{1}{z_n}} = y$ in $B_R(0)$.

(c) We will now show that the above cannot be true for a function $g(z)$ with a pole at $z = 0$. If g has a pole of order m at z_0, then we can write for z near z_0:

$$g(z) = \frac{h(z)}{(z - z_0)^m},$$

where $h(z)$ is holomorphic at z_0 and $h(z_0) \neq 0$. Therefore, we can easily calculate that

$$\lim_{z\to z_0} |g(z)| = \lim_{z\to z_0} \frac{|h(z)|}{|z - z_0|^m} = +\infty,$$

since $\lim_{z\to z_0} |h(z)| = |h(z_0)| \neq 0$ and $\lim_{z\to z_0} |z - z_0|^m = 0$. This means that, for any given $y \in \mathbb{C}$, we cannot have a sequence of points $z_n \xrightarrow{n\to+\infty} z_0$ solving $g(z_n) = y$; if that was the case, then we would have

$$|y| = \lim_{n\to+\infty} |g(z_n)| = +\infty \quad \text{since} \quad \lim_{z\to z_0} |g(z)| = +\infty,$$

which would be a contradiction. In particular, this means that there is a radius $R > 0$ such that no solution z of $g(z) = y$ lies in $|z - z_0| \leqslant R$ (because if the opposite was true, namely that the disc $B_R(z_0)$ contains a solution z_R of $g(z) = y$ for any $R > 0$, then as $R \to 0$ we could get a sequence of such solutions approaching z_0).